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Surface Impedance of Superconductors and Normal Conductors in EM Simulators[1]

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Introduction

Electromagnetic simulators can give very accurate solutions for microwave circuits with ideal conductors. When the conductors are non-ideal, accurate results may still be obtained in many cases by specifying material parameters or surface impedances. However, for structures in which the penetration depth of the field into the conductors is of the same order as the conductor thickness, considerable error can occur. This is not only a result of the conductor thickness being insufficient to contain the field completely, but is due in part to a separate effect which arises with some EM simulators when thick conductors are represented by thin sheets with surface impedance. For superconducting niobium microstrip circuits of typical dimensions, such errors can easily be as great as 20% in ε_{eff} and 10% in Z_0. In many cases, a simple correction to the surface impedance substantially improves the accuracy.

The concept of surface impedance

For an ideal conductor in an electromagnetic field, the tangential component E_t of the electric field at the surface is zero. A current flows in a thin sheet on the surface, as required to support the magnetic field H_t tangential to the surface. This short-circuit boundary condition excludes all fields from the interior of the ideal conductor. In a real conductor, fields extend into the conductor, but decrease rapidly with distance from the surface. To avoid the complication of solving Maxwell's equations inside conductors, it is usual to make use of the concept of surface impedance. The surface impedance $Z_S = E_t/H_t$ provides the boundary condition for fields outside the conductor, and accounts for the dissipation and energy stored inside the conductor.

For a thick plane conductor, the internal fields fall exponentially with distance from the surface, with 1/e depth Δ. For normal conductors, Δ is the classical skin depth $\delta = (2/\omega\sigma\mu)^{1/2}$, and $Z_S = (1+j)(\omega\mu/2\sigma)^{1/2}$. In Au or Cu at 100 GHz and room temperature, $\delta \approx 0.25$ µm, and $Z_S \approx 0.1(1+j)$ ohms/square. For a superconductor at a frequency well below its energy gap frequency, Δ is the London penetration depth, λ_L, which is independent of frequency. For niobium at ~4°K, at frequencies below ~700 GHz, $\lambda_L \approx 0.1$ µm. The surface impedance $Z_S = j\omega\mu_0\lambda_L$ ohms/square, corresponding to a surface inductance $L_S = \mu_0\lambda_L$ H/square, which is independent of frequency. In niobium, $L_S \approx 0.13$ pH/square, giving $Z_S \approx j0.08$ ohms/square at 100 GHz.

Two types of electromagnetic simulator

Two types of electromagnetic simulator are considered here: (i) finite-element solvers, such as HP/Ansof *hfss*, which divide the space between conductors into a three dimensional mesh and solve by matrix inversion for the fields at every mesh point, using the boundary conditions given by the surface impedance; and (ii) method-of-moments solvers, such as Sonnet *em,* which divide all conducting surfaces into (two dimensional) cells, and solve by matrix inversion for the currents in the cells, using the surface impedances as boundary conditions.

There is a subtle but fundamental difference between the solutions produced by the two types of simulator for circuits with thick conductors. In both cases fields inside the real conductors are taken into account by the surface impedance which provides the boundary conditions for the solution. This means that, in the simulation, the space corresponding to the interior of a conductor should be filled with a perfect magnetic conductor to constrain interior fields to zero. In the case of finite element solvers this is accomplished simply by terminating the spatial mesh at the conducting surfaces; *i.e.*, the mesh does not extend into the conductors. In the case of method-of-moments solvers there is no simple way to achieve the same result, and currents in the surface impedance do produce fields in the space "inside" the conductors if the surface impedance is not zero. It is therefore necessary to use a modified value of surface impedance when using method-of moments simulators for circuits with thick conductors. In many cases the correction is negligible, but in some cases (*e.g.*, superconducting microstrip transmission lines), it can be substantial.

Representation of conductors by surface impedances

To understand the way electromagnetic simulators treat a conductor of finite thickness, we examine the difference between an actual thick conductor and the model of the thick conductor which the simulator analyzes. The model of the conductor can be either a single thin sheet with the appropriate surface impedance, or a parallel pair of thin sheets separated by the thickness of the actual conductor.

Matick [1] has shown that the surface impedance seen by a plane wave normally incident on a conductor is the same as that seen by a wave traveling parallel to the conductor, as in a transmission line. For simplicity in the present discussion, we consider experiments in which a plane wave is normally incident on the surface of the conductor or model under test.

(a) A thick conductor represented as a single conducting sheet

Consider a plane wave normally incident on a plane (thick) conductor of surface impedance Z_S, as in Fig. 1(a). This is analogous to the circuit shown in Fig. 1(b), a long transmission line of characteristic impedance $Z_\eta = (\mu_0/\varepsilon_0)^{1/2} =$ 377 ohms, at whose end an impedance Z_S ohms is connected.

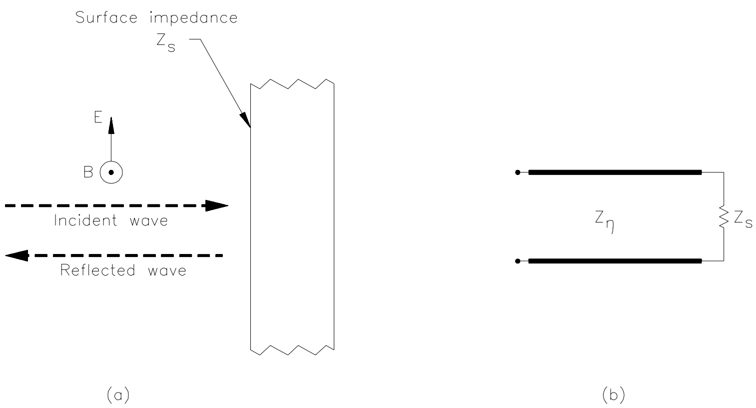

Figure 1

Next, consider a plane wave normally incident on a thin sheet of surface impedance Z_S, as in Fig. 2(a). The corresponding transmission line equivalent circuit is shown in Fig. 2(b) SS a long transmission line of characteristic impedance $Z_\eta = (\mu_0/\varepsilon_0)^{1/2} = 377$ ohms, at whose midpoint A an impedance Z_S ohms is connected in parallel. With the plane wave incident from the left, the field on the line to the right of A is zero only if $Z_S = 0$. At A, the incident wave sees an impedance Z_S in parallel with 377 ohms (the right half of the long transmission line), as in Fig. 2(c).

Clearly, the thin sheet with surface impedance Z_S (Fig. 1) is not physically equivalent to a (thick) conductor of surface impedance Z_S (Fig. 2) unless $Z_S = 0$. The apparent surface impedance, seen by the incident plane-wave, in Fig. 2 is Z_S in parallel with 377 ohms, and some power is coupled through the thin sheet into the space on the other side. For cases in which $|Z_S| \ll 377$ ohms/square (*i.e.*, most practical cases), the error is inconsequential.

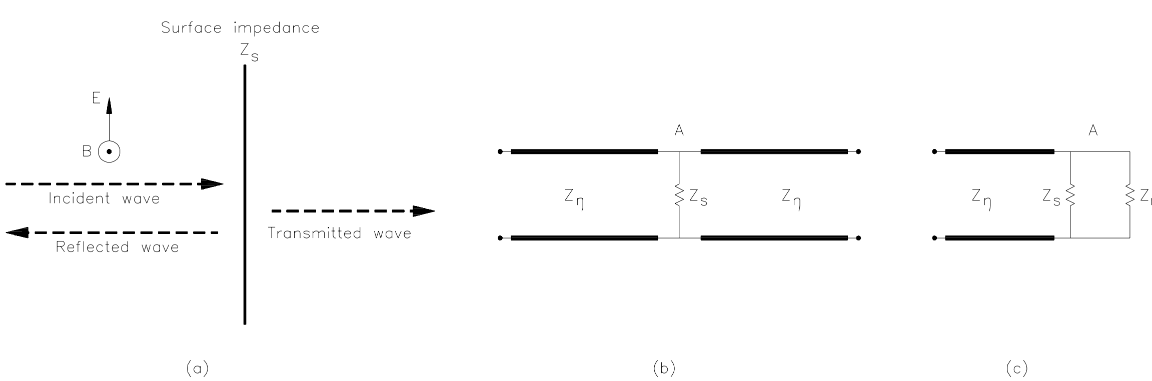

Figure 2

(b) A conductor of thickness t represented as a pair of conducting sheets

Consider the reflection of a plane wave from a conductor of thickness t, as shown in Fig. 3(a). The incident wave sees an impedance Z_S at the surface of the conductor - the value of Z_S is not the same as in the previous example. The impedance seen by the incident wave is as depicted in the circuit of Fig. 3(b). The appropriate value of Z_S for finite values of t is given in a later section.

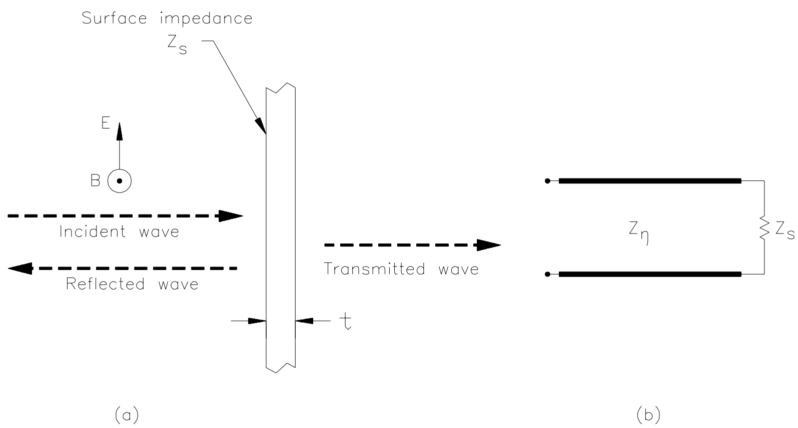

Figure 3

Now consider a plane wave normally incident on a pair of thin sheets, of surface impedance Z_S, separated by distance t, as in Fig. 4(a). This is analogous to the circuit shown in Fig. 4(b), a long transmission line of characteristic impedance $Z_\eta = (\mu_0/\varepsilon_0)^{1/2} = 377$ ohms, at whose midpoint A an impedance Zs is connected, with a second impedance Zs a distance t to the right. If the distance t is much less than the wavelength, the impedance seen by a plane wave incident from the left is as depicted in Fig. 4(c). The inductance $L = \mu_0 t$ accounts for the energy stored in the magnetic field between the conducting sheets. For a conductor 0.3 μm thick, at 100 GHz, the reactance $\omega L = \omega\mu_0 t = 0.24$ ohms/square.

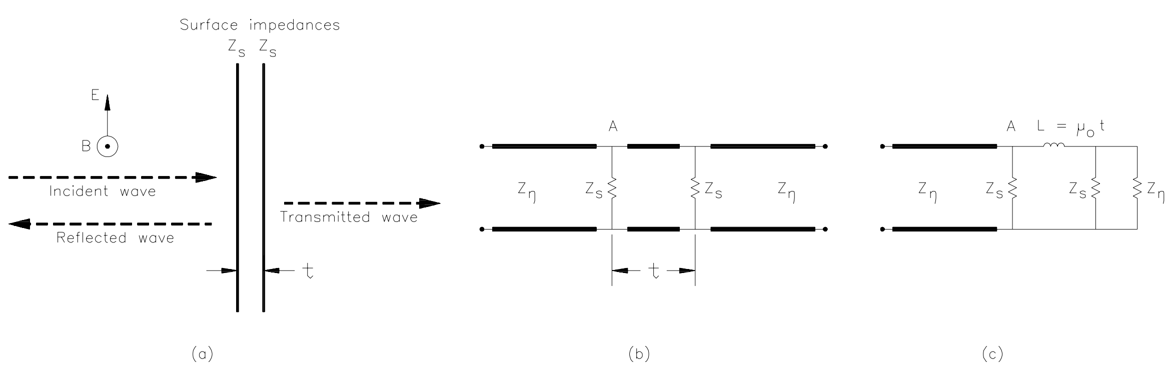

Figure 4

It is clear that if a conductor is thick enough, $\omega\mu_0 t \gg Z_S$, and the two-sheet representation is sufficiently accurate. For normal metal conductors, this requires that $t \gg \delta/2$, and for superconductors $t \gg \lambda_L$.

Surface impedance of conductors of finite thickness

(a) Excitation from one side

When the thickness t of a conductor is not very much greater than the penetration depth Δ, a field on one side of the conductor penetrates partially through to the other side. For normal conductors the surface impedance seen by the incident field is (see Appendix):

$$Z_S = \frac{k}{\sigma}\,\frac{e^{kt} + \dfrac{\sigma Z_\eta - k}{\sigma Z_\eta + k}\,e^{-kt}}{e^{kt} - \dfrac{\sigma Z_\eta - k}{\sigma Z_\eta + k}\,e^{-kt}}\,.$$

Here $k = (1 + j)/\delta$, and $Z_\eta = (\mu/\varepsilon)^{1/2}$ is the characteristic impedance of space (377 ohms in vacuum).

In most cases $Z_\eta \gg k/\sigma$, and $Z_S = \dfrac{k}{\sigma}\,\dfrac{e^{kt} + e^{-kt}}{e^{kt} - e^{-kt}}$. When t is large, this reduces to the usual surface impedance formula $Z_S = (1+j)(\omega\mu/2\sigma)^{1/2}$.

In the case of a superconductor, when the thickness t is not much greater than the London penetration depth λ_L, the surface impedance is (see Appendix):

$$Z_S = j\omega\mu\lambda_L\,\frac{e^{\frac{t}{\lambda_L}} + \dfrac{Z_\eta - j\omega\mu\lambda_L}{Z_\eta + j\omega\mu\lambda_L}\,e^{-\frac{t}{\lambda_L}}}{e^{\frac{t}{\lambda_L}} - \dfrac{Z_\eta - j\omega\mu\lambda_L}{Z_\eta + j\omega\mu\lambda_L}\,e^{-\frac{t}{\lambda_L}}}\,,$$

where again $Z_\eta = (\mu/\varepsilon)^{1/2}$ is the characteristic impedance of space (377 ohms in vacuum). In most cases $Z_\eta \gg \omega\mu\lambda_L$, so $Z_S = j\omega\mu\lambda_L \coth t/\lambda_L$. When $t \gg \lambda_L$ this becomes the usual formula for superconductors: $Z_S = j\omega\mu_0\lambda_L$.

(b) Symmetric and anti-symmetric excitation from both sides

In the above, it has been assumed that the field is incident on the conductor from one side only. This is the case for ground planes, waveguide walls, wide parallel-plate transmission lines, and wide microstrip lines. In cases such as a stripline center conductor, equal fields are present on *both* sides of the conductor. In a few cases, such as a septum across a waveguide (parallel to the broad walls), equal but *opposite* fields are present on the two sides. For not-very-thick conductors in such symmetrical or anti-symmetrical fields, the effective surface impedance seen from one side is modified by the presence of the field on the other.

For a normal metal conductor of thickness t with symmetrical or anti-symmetrical excitation, the surface impedance is (see Appendix):

$$Z_S = \frac{k}{\sigma}\left[\frac{e^{kt} + e^{-kt}}{e^{kt} - e^{-kt}} \pm \frac{2}{e^{kt} - e^{-kt}}\right],$$

where $k = (1 + j)/\delta$. The + sign is for symmetrical fields on the two sides, and the - sign for anti-symmetrical fields.

For a superconductor of thickness t with symmetrical or anti-symmetrical excitation, the surface impedance is (see Appendix):

$$Z_S = j\omega\mu\lambda_L\left[\coth\frac{t}{\lambda_L} \pm \mathrm{csech}\frac{t}{\lambda_L}\right].$$

Again, the + sign is for symmetrical fields on the two sides, and the - sign for anti-symmetrical fields. The following table gives the values of the coth and sinh terms, and their sum, for typical Nb conductor thicknesses assuming $\lambda_L = 1000$ Å.

t Å	t/λ	coth(t/λ)	csech(t/λ)	coth(t/λ) + csech(t/λ)
5000	5.0	1.000	0.013	1.014
4000	4.0	1.001	0.037	1.037
3000	3.0	1.005	0.100	1.105
2500	2.5	1.014	0.165	1.179
2000	2.0	1.037	0.276	1.313
1500	1.5	1.105	0.470	1.574
1200	1.2	1.200	0.662	1.862
1000	1.0	1.313	0.851	2.164
800	0.8	1.506	1.126	2.632

Modified surface impedance for thin conducting sheets representing a thick conductor.

A modified value of surface impedance can be used to correct the discrepancy between the real conductor and the two-sheet model. Let Z_S be the desired surface impedance as given by the appropriate formula above, and let Z_X be the value of surface impedance of the conducting sheets which will result in an effective surface impedance of Z_S as seen by an incident wave, as depicted in Fig. 5.

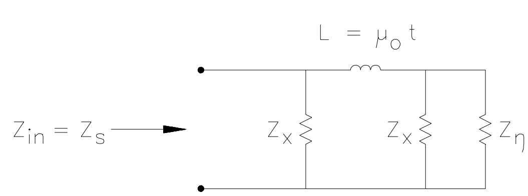

Figure 5

In most practical cases Z_η is large compared with the other circuit elements, and can be ignored. Then, analysis of the circuit gives a quadratic equation in Z_X whose solution is

$$Z_X = \frac{1}{2}\left[\left(2Z_S - j\omega\mu_0 t\right) \pm \left[4Z_S^2 + \left(j\omega\mu_0 t\right)^2\right]^{\frac{1}{2}}\right].$$

In the case of a superconductor excited from one side, $Z_S = j\omega\mu_0\lambda_L \coth(t/\lambda_L)$. It follows that $Z_x = \beta Z_S$, where

$$\beta = \left[1 - \frac{t}{2\,\lambda_L \coth\frac{t}{\lambda_L}} + \sqrt{1 + \left(\frac{t}{2\,\lambda_L \coth\frac{t}{\lambda_L}}\right)^2}\right].$$

Fig. 6 shows β as a function of t/λ_L.

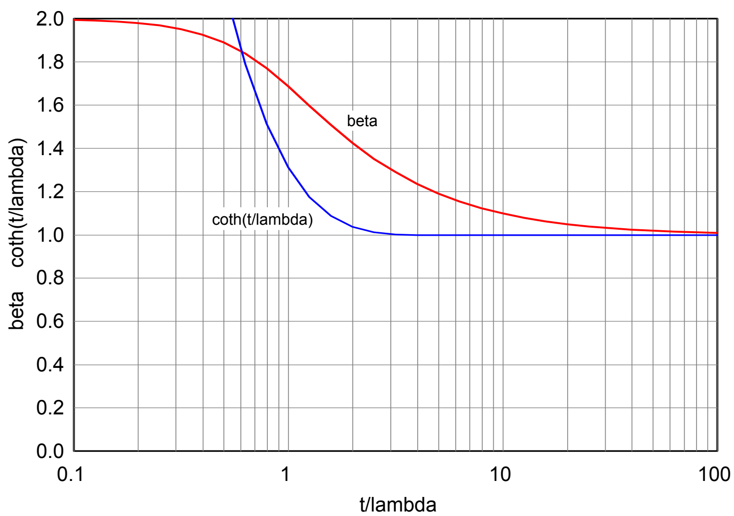

Figure 6

Examples

To demonstrate the significance of the β and coth corrections to the surface impedance, consider a superconducting Nb microstrip transmission line of width 6 µm, with a 0.2 µm-thick dielectric layer with $\varepsilon_r = 3.8$, over a Nb ground plane. The London penetration depth $\lambda_L = 0.1$ um. In the first example, the Nb conductors are 0.1 µm thick, and in the second example they are 0.3 µm thick. The table below gives the effective dielectric constant and characteristic impedance of the microstrip when the upper conductor is represented by a pair of conducting sheets. Sonnet *em* was used, with the thick-conductor value of the surface impedance Z_S and the following corrections: (i) both β and $\coth(t/\lambda_L)$ corrections, (ii) only the $\coth(t/\lambda_L)$ correction, and (iii) no corrections. Corresponding results are also shown for the same microstrips (iv) with the upper conductor characterized as a single conducting sheet whose surface impedance includes the coth correction (but not the β correction, which applies only when two sheets are used), and, (v) with perfect conductors ($Z_S = 0$). The second table gives the same results expressed as percentage deviations from the most accurete solution, (i).

	Nb thickness = 0.1 µm		Nb thickness = 0.3 µm	
	ε_{eff}	Z_0	ε_{eff}	Z_0
(i) Coth & β corrections	8.32	8.75	6.95	8.13
(ii) Coth correction only	7.25	8.30	6.55	7.92
(iii) No coth or β corrections	6.41	7.84	6.53	7.91
(iv) Single-sheet	8.28	9.04	7.19	8.42
(v) Perfect conductors	3.55	5.90	3.53	5.87

	Nb thickness = 0.1 µm % errors wrt top line		Nb thickness = 0.3 µm % errors wrt top line	
	ε_{eff}	Z_0	ε_{eff}	Z_0
(i) Coth & β corrections	0%	0%	0%	0%
(ii) Coth correction only	-13%	-5%	-6%	-3%
(iii) No coth or β corrections	-23%	-10%	-6%	-3%
(iv) Single-sheet	-1%	3%	3%	4%
(v) Perfect conductors	-57%	-33%	-49%	-28%

It is also of interest to compare the results obtained by Sonnet *em* with the most accurate analytical results available. We use the analytical results from a recent report by Yassin & Withington [2] for Nb microstrip lines of width 2, 4, and 6 µm, with a 0.3 µm dielectric layer of $\varepsilon_r = 3.8$, with a Nb groundplane. The center conductor and groundplane are 0.3 µm thick, and $\lambda_L = 0.1$ µm, so $t/\lambda_L = 3$. For this value of t/λ_L, the β correction is significant, but the coth correction is very small. The results for the effective dielectric constant and characteristic impedance are compared below. Agreement is very close, except for the narrowest line, in which case there is a 4% disagreement in Z_0.

	Microstrip width 2 µm		Microstrip width 4 µm		Microstrip width 6 µm	
	ε_{eff}	Z_0	ε_{eff}	Z_0	ε_{eff}	Z_0
Ref. [2]	5.13	26.1	5.50	15.1	5.68	10.7
Sonnet em	5.19	27.2	5.52	15.4	5.70	10.8
% difference	1%	4%	0%	2%	0%	1%

Acknowledgment

The author thanks Jim Merrill of Sonnet Software for his helpful discussions on the role of surface impedance in Sonnet *em*. He also thanks S.-K. Pan of NRAO for pointing out some errors in the earlier version.

References

[1] R.E. Matick, "Transmission Lines for Digital and Communication Networks", New York: McGraw-Hill, 1969.

[2] G. Yassin and S. Withington, "Electromagnetic models for superconducting millimeter-wave and submillimeter-wave microstrip transmission line," Journal of Physics D: Applied Physics, vol. 28, no. 9, pp. 1983-1991, 14 September 1995.

APPENDIX: Derivation of formulas

§A1 Inductance of a thin layer containing a uniform magnetic field

If a plane wave, normally incident on a perfect plane conductor, produces a current J A/m in the conductor, then by Ampere's law, the magnetic field near the conductor $B = J\mu$. In a layer of thickness dx parallel to the conductor, the stored magnetic energy $dW = B^2dx/2\mu = J^2\mu dx/2$ per unit area. Let the inductance contributed by the magnetic field in this layer be dL H/square. This inductance is in series with the current J A/m. Then the energy stored in this inductance is $J^2dL/2$ per unit area. It follows that $dL = \mu dx$ H/square.

§A2 Surface impedance Z_S and skin depth δ of a normal conductor

Consider a plane wave incident on a thick conductor. The incident wave excites voltages and currents in the conductor which vary with depth from the surface. An incremental thickness dx of a unit area of the conductor is characterized by the equivalent circuit of Fig. A1. From §A1 above, the magnetic field in the volume of thickness dx accounts for a series inductance μdx H/square. The conductivity σ has a parallel conductance σdx S/square. Hence $dZ = j\omega\mu dx$ and $dG = \sigma dx$. For this circuit, the input impedance is the surface impedance Z_S.

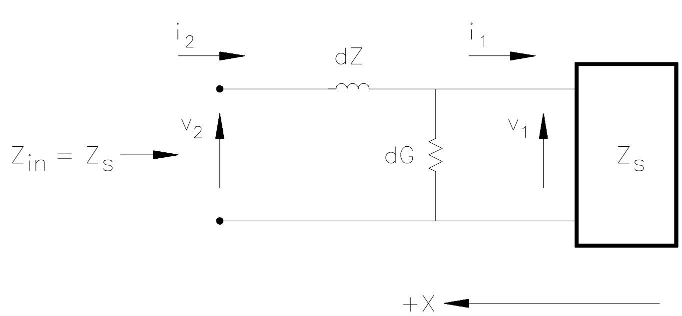

Fig. A1

Since the conductor is thick, the impedance looking to the right at any depth in the conductor is equal to the surface impedance Z_S. Hence

$$Z_{in} = Z_S = dZ + \frac{1}{dG + \frac{1}{Z_S}} = j\omega\mu\,dx + \frac{1}{\sigma\,dx + \frac{1}{Z_S}} \; .$$

Solving for Z_S gives $Z_S^2 = j\omega\mu/\sigma$, whence the standard result:

$$\boxed{Z_S = (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}} \; .}$$

From the figure, $di = i_2 - i_1 = v_1\,dG = i_1 Z_S\,dG = i_1 Z_S \sigma\,dx$.

Therefore $$\int_{i_0}^{i} \frac{di}{i} = Z_S\,\sigma\,dx \qquad \text{or} \qquad i = i_0\, e^{Z_S \sigma (x - x_0)} \; .$$

The sign of the exponent is positive because of the choice of x-direction in Fig. A1. With the above expression for Z_S,

$$i = i_0\, e^{\sqrt{\frac{\omega\sigma\mu}{2}}(x - x_0)} . e^{j\sqrt{\frac{\omega\sigma\mu}{2}}(x - x_0)} ,$$

from which the skin depth is $$\boxed{\delta = \sqrt{\frac{2}{\omega\sigma\mu}} \; .}$$

§A3 Surface impedance Z_S and penetration depth λ_L of a superconductor

The analysis for a superconductor is similar to that for a normal conductor, with the exception that the conductance element dG is replaced by a susceptance. As the superconductor is lossless, the current is limited only by the inertia of the Cooper pairs of electrons, which manifests itself as a kinetic inductance.

Consider a layer of superconductor of thickness dx. In terms of the average velocity of the carriers S, the current di = $(n^* e^* S)dx$ A/m, where n^* is the effective density of carriers with effective charge e^*. If an AC voltage $v = Ve^{j\omega t}$ V/m is applied parallel to the surface, the force on a carrier is $e^* Ve^{j\omega t} = m^* dv/dt$, where m^* is the effective mass of a carrier. The carrier velocity

$$S = \frac{e^*}{m^*}\int V e^{j\omega t}\,dt = \frac{1}{j\omega}\frac{e^*}{m^*} V e^{j\omega t} \; .$$

The current in the layer of thickness dx is therefore

$$di \;=\; \frac{1}{j\omega}\,\frac{n^{*}e^{*2}}{m^{*}}\,V\,e^{j\omega t}\,dx \qquad \text{A/m.}$$

Writing $di = dIe^{j\omega t}$ gives $\quad V \;=\; j\omega\,\frac{m^{*}}{n^{*}e^{*2}}\,\frac{1}{dx}\,dI\ ,$

from which it is evident that the kinetic inductance of the layer is given by

$$d\left(\frac{1}{L}\right) \;=\; \frac{n^{*}e^{*2}}{m^{*}}\,dx\ .$$

Now consider a plane wave incident on a thick superconductor. The incident wave excites voltages and currents which vary with depth from the surface. An incremental thickness dx of a unit area of the superconductor is characterized by the equivalent circuit of Fig. A2.

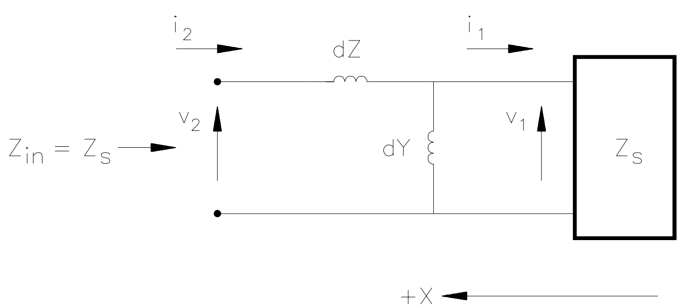

Fig. A2

From §A1 above, the magnetic field in the volume of thickness dx accounts for a series inductance μdx H/square. Hence $dZ = j\omega\mu dx$. The kinetic inductance of the Cooper pairs in the same volume contributes a parallel admittance

$$dY \;=\; \frac{1}{j\omega}\,d\left(\frac{1}{L}\right) \;=\; \frac{n^{*}e^{*2}}{j\omega\, m^{*}}\,dx\ .$$

Since the conductor is thick, the impedance looking to the right at any depth in the conductor is equal to the surface impedance Z_S. Hence, in Fig. A2,

$$Z_{in} \;=\; Z_S \;=\; dZ + \frac{1}{dY + \dfrac{1}{Z_S}} \;=\; j\omega\mu\,dx + \frac{1}{\dfrac{n^{*}e^{*2}}{j\omega\, m^{*}}\,dx + \dfrac{1}{Z_S}}\ .$$

Solving for Z_S gives $\quad \boxed{Z_S \;=\; j\omega\sqrt{\dfrac{\mu\, m^{*}}{n^{*}e^{*2}}}\ .}$

To deduce the penetration depth in a superconductor, consider again the circuit of Fig. A2:

$$di = i_2 - i_1 = v_1\, dY = i_1 Z_S\, dY \ .$$

With the above expressions for dY and Z_S,

$$di = \sqrt{\frac{\mu\, n^* e^{*2}}{m^*}}\ i\, dx \ .$$

$$\therefore \quad i = i_0\, e^{\sqrt{\frac{\mu\, n^* e^{*2}}{m^*}}\,(x - x_0)} \ .$$

$$\text{or} \quad i = i_0\, e^{\frac{(x - x_0)}{\lambda_L}} \ .$$

(The sign of the exponent is positive because of the choice of x-direction in Fig. A2).

The quantity $\boxed{\lambda_L = \sqrt{\frac{m^*}{\mu\, n^* e^{*2}}}}$ is the London penetration depth, and is independent of frequency.

The expression for the surface impedance can be written in terms of λ_L as $\boxed{Z_S = j\omega\,\mu\,\lambda_L}$ ohms/square, which corresponds to a surface inductance $\boxed{L_S = \mu\,\lambda_L}$ H/square.

<u>§A4 Surface impedance Z_S of a normal conductor of finite thickness</u>

To deduce the surface impedance of a normal conductor of thickness t, consider first an incremental thickness dx of the conductor. This is represented by the equivalent circuit of Fig. A3.

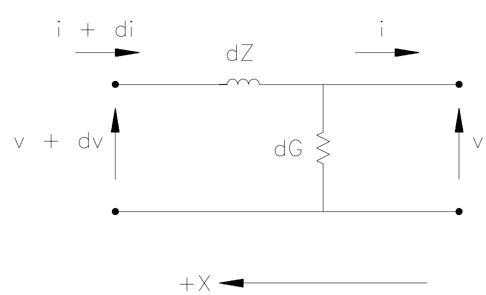

Fig. A3

In the figure, $$di = v\, dG = \sigma\, v\, dx$$

and $$dv = i\, dZ = j\omega\,\mu\, i\, dx \ ,$$

therefore $$\frac{d^2 i}{dx^2} = j\omega\sigma\mu i .$$

This has the solution $$i = i_+ e^{kx} + i_- e^{-kx},$$

where $$k = \sqrt{j\omega\sigma\mu} = (1 + j)\sqrt{\frac{\omega\sigma\mu}{2}} = \frac{1 + j}{\delta} ,$$

and δ is the classical skin depth as derived above.

Now consider the equivalent circuit of the conductor, terminated on the right by the impedance of space Z_η, as shown in Fig. A4.

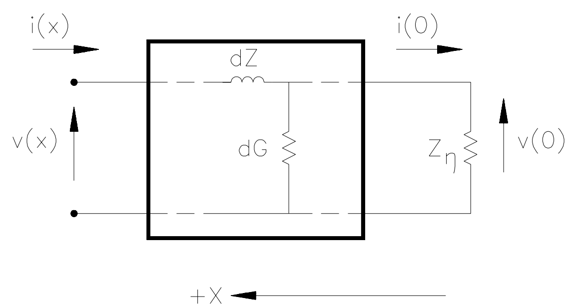

Fig. A4

In Fig. A4, $$i = i_+ e^{kx} + i_- e^{-kx}$$

and $$v = \frac{1}{\sigma}\frac{di}{dx} = \frac{k}{\sigma}(i_+ e^{kx} - i_- e^{-kx}).$$

At x = 0, $$\frac{v}{i} = \frac{k}{\sigma}\frac{i_+ - i_-}{i_+ + i_-} = Z_\eta .$$

Therefore $$i_- = -i_+ . \frac{\sigma Z_\eta - k}{\sigma Z_\eta + k} ,$$

and hence,
$$Z_S = \frac{v(t)}{i(t)} = \frac{k}{\sigma}\left[\frac{e^{kt} + \dfrac{\sigma Z_\eta - k}{\sigma Z_\eta + k} e^{-kt}}{e^{kt} - \dfrac{\sigma Z_\eta - k}{\sigma Z_\eta + k} e^{-kt}}\right],$$

where
$$k = \sqrt{j\omega\,\sigma\,\mu} = (1 + j)\sqrt{\frac{\omega\,\sigma\,\mu}{2}} = \frac{1 + j}{\delta}\ .$$

In most practical situations $Z_\eta \gg \frac{k}{\sigma}$, so
$$Z_S \quad \frac{k}{\sigma}\,\frac{e^{kt} + e^{-kt}}{e^{kt} - e^{-kt}}\ .$$

§A5 Surface impedance Z_S of a superconductor of finite thickness

The analysis in this case follows that for the normal conductor but with dG replaced with $dY = \frac{n^* e^{*2}}{j\omega\, m^*}\,dx$. It follows that $\frac{d^2 i}{dx^2} = \frac{1}{\lambda_L^2}\, i$, and
$$Z_S = \frac{v(t)}{i(t)} = j\omega\,\mu\,\lambda_L\left[\frac{e^{\frac{t}{\lambda_L}} + \dfrac{Z_\eta - j\omega\,\mu\,\lambda_L}{Z_\eta + j\omega\,\mu\,\lambda_L}\, e^{-\frac{t}{\lambda_L}}}{e^{\frac{t}{\lambda_L}} - \dfrac{Z_\eta - j\omega\,\mu\,\lambda_L}{Z_\eta + j\omega\,\mu\,\lambda_L}\, e^{-\frac{t}{\lambda_L}}}\right],$$

where $\lambda_L = \sqrt{\frac{m^*}{\mu\, n^* e^{*2}}}$ is the London penetration depth derived above. Usually, $Z_\eta \gg j\omega\mu\lambda_L$, in which case we obtain the usual formula:
$$Z_S \quad j\omega\,\mu\,\lambda_L \coth\frac{t}{\lambda_L}\ .$$

§A6 Effective surface impedance Z_S of a normal conductor of finite thickness excited from both sides

When a conductor of finite thickness has fields incident on both sides, the apparent surface impedance on either side is affected by the field on the other. From above, and with reference to Fig. A4: When the excitation is on one side only,

$$i = i_+ e^{kx} + i_- e^{-kx}$$

and
$$v = \frac{k}{\sigma}(i_+ e^{kx} - i_- e^{-kx}),$$

where
$$k = \sqrt{j\omega\sigma\mu} = (1 + j)\sqrt{\frac{\omega\sigma\mu}{2}} = \frac{1 + j}{\delta}.$$

For one-sided excitation, and assuming $Z_\eta >> k/\sigma$: At x = 0, i = 0, so $i_- = -i_+$.

Therefore,
$$v(0) = \frac{k}{\sigma} i_+.(e^{kx} + e^{-kx}) = 2\frac{k}{\sigma} i_+.$$

At x = t
$$i(t) = i_+.(e^{kx} - e^{-kx})$$

and
$$v(t) = \frac{k}{\sigma} i_+.(e^{kx} + e^{-kx}).$$

When the circuit is excited by equal current sources $i = i_+.(e^{kt} - e^{-kt})$ on both sides, then at x = t, using superposition:

$$v(t) = \frac{k}{\sigma} i_+.(e^{kt} + e^{-kt}) + 2\frac{k}{\sigma} i_+.$$

Therefore,
$$\boxed{Z_S = \frac{v(t)}{i(t)} = \frac{k}{\sigma}\left[\frac{e^{kt} + e^{-kt}}{e^{kt} - e^{-kt}} + \frac{2}{e^{kt} - e^{-kt}}\right]}.$$

If the excitation on the two sides is out of phase, the sign of the second term in the square brackets becomes negative.

§A7 Effective surface impedance Z_S of a superconductor of finite thickness excited from both sides

The approach follows that used above for the normal conductor. For single-sided excitation, referring to Fig. A4,

$$i = i_+ e^{\frac{x}{\lambda_L}} + i_- e^{-\frac{x}{\lambda_L}}$$

and
$$v = j\omega\mu\lambda_L (i_+ e^{\frac{x}{\lambda_L}} - i_- e^{-\frac{x}{\lambda_L}}).$$

For one-sided excitation, and assuming $Z_\eta >> \omega\mu\lambda_L$: At x = 0, i = 0, so $i_- = -i_+$.

Therefore, $$v(0) = j\omega\,\mu\,\lambda_L\, i_+.(e^{\frac{x}{\lambda_L}} + e^{-\frac{x}{\lambda_L}}) = 2\,j\omega\,\mu\,\lambda_L\, i_+ .$$

At x = t $$i(t) = i_+.(e^{\frac{t}{\lambda_L}} - e^{-\frac{t}{\lambda_L}})$$

and $$v(t) = j\omega\,\mu\,\lambda_L\, i_+.(e^{\frac{t}{\lambda_L}} + e^{-\frac{t}{\lambda_L}}).$$

When the circuit is excited by equal current sources $i = i_+.(e^{kt} - e^{-kt})$ on both sides, then at x = t, using superposition:

$$v(t) = j\omega\,\mu\,\lambda_L\, i_+.(e^{\frac{t}{\lambda_L}} + e^{-\frac{t}{\lambda_L}}) + 2\,j\omega\,\mu\,\lambda_L\, i_+ .$$

Therefore, $$\boxed{Z_S = \frac{v(t)}{i(t)} = j\omega\,\mu\,\lambda_L\left[\coth\frac{t}{\lambda_L} + \operatorname{csech}\frac{t}{\lambda_L}\right]}.$$

If the excitation on the two sides is out of phase, the sign of the second term in the square brackets becomes negative.