MMA Memo 255

Fine Adjustment in the MMA Delay System.

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Abstract The main requirements for fine adjustment of the variable delays required for the MMA are considered. A system using delay increments of 1/16 of the Nyquist sampling interval, as used for the VLA, would be a good solution. The rate of delay adjustment would be approximately once every 32 ms for a maximum east-west baseline of 2 km or 6.4 msec for 10 km. Reduction in sensitivity and errors in the visibility phase are considered. The possibility of compensating for the delay errors after correlation, as in most VLBI arrays, is also considered. With the latter method, operation in the phased-array mode suffers a sensitivity loss of approximately 13% (in voltage) with the full 16 GHz bandwidth, but becomes acceptable if the phased-mode bandwidth is no more than 8 GHz.

In a synthesis array it is necessary to provide adjustable instrumental delays for the signals from each antenna to compensate for the changes in the geometrical path lengths to the antennas for an incoming wavefront, as the antennas track a source across the sky. The signals are sampled at the Nyquist rate before reaching the correlator. These samples are spaced at time intervals of $1/2\Delta\nu$ which, if used by itself for delay adjustment, results in rather too much loss in sensitivity. In the VLA, for example, further adjustment is provided by shifting the phase of the sampler clock in increments of 1/16 of a cycle which provides timing adjustment in increments of $1/32\Delta\nu$, where $\Delta\nu$ is the bandwidth. In VLBI, correction for fine delay errors is commonly made after correlation.

Delay Adjustment by Sampler Clock Phasing

Consider a minimum delay increment τ_0. If the delay is adjusted by one increment every time the magnitude of the delay error reaches $\tau_0/2$, then the probability distribution for the delay error for any one antenna takes the form of a rectangular (constant amplitude) function of width τ_0 centered on zero error. In general the rate of change of the geometrical delay will be different for each antenna, so for any pair of antennas the times at which the delay adjustments occur will be unrelated. Thus the probability distribution of the combined error for any antenna pair will be a triangular function of width $2\tau_0$ centered on zero. The rms delay error for a pair of antennas is then $\tau_0/\sqrt{6}$. The corresponding phase errors in the visibility values depend upon the signal frequency at which the delay is inserted. The baseband IF that is sampled extends from approximately zero to the IF bandwidth $\Delta\nu$. The rms value of the frequency for this range is $\Delta\nu/\sqrt{3}$. For continuum observations the rms value of the phase errors is the product of the rms values of the delay errors and the baseband frequency which is

$$\Delta\phi_{\rm rms} = \frac{2\pi\Delta\nu\tau_0}{3\sqrt{2}} \ \ \text{radians}. \tag{1}$$

For spectral line observations the frequency of the spectral channels within the baseband varies from approximately zero to $\Delta\nu$. Thus in the worst case the rms phase errors are those for the highest baseband-frequency channel and are $\sqrt{3}$ greater than in expression (1).

To determine the loss in sensitivity resulting from phase errors, the signals from two antennas can be represented by $V_1 e^{\phi_1}$ and $V_2 e^{\phi_2}$ at the correlator inputs, where the ϕ terms are the phase errors. The correlator output is

$$R = \langle V_1 e^{\phi_1} V_2^* e^{\phi_2} \rangle, \tag{2}$$

where the brackets $\langle\rangle$ represent time averaging. Then if $\Delta\phi = (\phi_1 - \phi_2)$ is the phase error we have

$$R = V_1 V_2^* [\langle \cos\Delta\phi \rangle + j\langle \sin\Delta\phi \rangle]. \tag{3}$$

Since the probability distribution of $\Delta\phi$ is an even function with zero mean the time average of the sine term has an expectation of zero. If the phase errors are small so that fourth-power terms in $\Delta\phi$ can be neglected, we have

$$R = V_1 V_2^* [1 - \tfrac{1}{2}\langle \Delta\phi^2 \rangle]. \tag{4}$$

From expression (1) we obtain for the mean squared phase errors

$$\langle \Delta\phi^2 \rangle = \frac{2(\pi\Delta\nu\tau_0)^2}{9} \ \ \text{rad}^2. \tag{5}$$

Thus the sensitivity is reduced by a factor

$$[1 - (\pi\Delta\nu\tau_0)^2/9]. \tag{6}$$

For the VLA, $\tau_0 = 1/32\Delta\nu$, that is, the timing of the sampler pulses can be varied in increments of 1/16 of the sample interval. Thus the rms phase error resulting from the finite increments in delay setting is $\pi/(48\sqrt{2}) =$

4.63×10^{-2} radians or 2.65°. The corresponding loss in sensitivity, i.e., loss in amplitude, is 0.11%. Phase errors resulting from the delay errors depend upon the number of times that the delay is reset during the averaging time at the correlator output. This number could be as high as a few hundred for the longest baselines, and for such cases the rms phase error in the visibility can be less than that of the individual delay-induced errors by an order of magnitude or more. However, the rate of change of delay with time is proportional to the east-west component of the baseline as projected onto the sky, which goes through zero as the baseline vector crosses the v axis. In the worst case the visibility error is then the product of the baseband frequency and the maximum delay error of τ_0, can which occur if the delay errors of the two antennas are of magnitude $\tau_0/2$ and of opposite sign. (Although the probability of occurrence of the delay error goes to zero at τ_0, the probability becomes significant for slightly smaller errors since the error distribution is triangular, not Gaussian.) For continuum observations the corresponding visibility error is equal to τ_0 multiplied by the mean intermediate frequency $\Delta\nu/2$, which for the VLA is $\pi/32$ rad, or 5.6°. For spectral line observations the phase error varies with the baseband frequency corresponding to the spectral channel, and for the high end of the baseband can be twice the error for continuum, i.e., 11.2°. The table below gives the rms phase error (before averaging), the loss in sensitivity from (6), and the maximum phase error for a spectral line channel at the high end of the baseband. The top line corresponds to the VLA system.

$\tau_0\Delta\nu$	RMS Phase Error	Loss in Sensitivity	Maximum Phase error
1/32	2.65°	0.11%	11.2°
1/16	5.3°	0.43%	22.4°
1/8	10.6°	1.71%	44.8°

The loss in sensitivity is certainly acceptably small for $\tau_0\Delta\nu$ equal to 1/32 and 1/16, and possibly also for 1/8. The maximum visibility phase errors are more serious, although in principle they could be corrected after averaging since the delay settings are known. This would only need to be done for cases where the number of delay settings within the averaging interval is small. However, this is hardly necessary for $\tau_0\Delta\nu = 1/32$, and this would be a good choice for the MMA.

The rate of change of delay for a pair of antennas is equal to $\omega_e x_{EW} \cos\delta/c$ where $\omega_e = 7.29 \times 10^{-5}$ rad/s is the angular velocity of the earth, x_{EW} is the east-west component of the baseline (equal to u times the wavelength), δ is the declination of the phase reference position, and c is the velocity of light. For the MMA (original specification) the longest east-west baseline component is 2 km. The corresponding rate of change of delay for the celestial equator is 4.86×10^{-10} (seconds of delay per second of UT). For the maximum MMA baseband bandwidth of 2 GHz and $\tau_0\Delta\nu = 1/32$ (for which $\tau_0 = 1.56 \times 10^{-11}$), the corresponding delay adjustment rate is once every 32.1 msec. With the combined MMA and LSA the longest baselines might be $\sim$ 10 km, in which case the corresponding maximum rate of delay adjustment is once every 6.4 msec. However, in an array each antenna's delay is adjusted individually with respect to a common reference position on the ground. If the reference position is chosen to be the center of the array, then each of the two antennas of a 2 km east-west pair would be adjusted at a rate corresponding to a 1 km baseline, so the actual rates of change of delay could be half those in the example just given.

Delay Adjustment After Correlation

Instead of adjusting the phase of the sampler clock to obtain the fine delay steps it is possible to apply a correction for delay errors after cross correlation. This is the method commonly used in VLBI. The correlated data are transformed to the frequency domain and the phases are incremented linearly with frequency to correct for the effect of the variation in the geometrical time delay. (In VLBI this procedure is known as the fractional bit-shift correction, see, e.g., TMS pp. 302–303). The correlator output has to be dumped at regular intervals to allow for this procedure. It must be dumped rapidly enough that the signal is not significantly reduced by averaging over the changing phase.

Over a short time period the phase at the correlator output varies linearly with time as a result of the changing geometric delay. Let τ_d be the increment in the geometric delay between correlator dumps. The effective delay error over this period has a probability distribution that is uniform from $-\tau_d/2$ to $\tau_d/2$, and the rms error is $\tau_d/(2\sqrt{3})$. For a frequency ν' within the baseband the corresponding rms phase error is

$$\Delta\phi_{\mathrm{rms}} = \pi\tau_d\nu'/\sqrt{3}. \tag{7}$$

For continuum observations we should use the rms value of ν' over the bandwidth $\Delta\nu$, which is $\Delta\nu/\sqrt{3}$. Let $X\%$ be the loss in continuum sensitivity that can be tolerated. From (4), $\Delta\phi_{\mathrm{rms}} = \sqrt{2X}/10$. So we have

$$\tau_d = \frac{3\sqrt{2X}}{10\pi\Delta\nu}. \tag{8}$$

Now suppose that we can accept a loss of $X = 1\%$. Then with $\Delta\nu = 2$ GHz we obtain $\tau_d = 6.75 \times 10^{-11}$ s.

As calculated earlier, the rate of change of delay for a 2-km, east-west baseline $4.86{\times}10^{-10}$. The corresponding correlator dump interval for $\tau_d = 6.75 \times 10^{-11}$ s is 139 ms. For a 10-km, east-west baseline the dump interval is 28 ms. For spectral line observations the sensitivity loss varies with the baseband frequency. For the highest frequency channel we should use $\nu' = \Delta\nu$ (instead of $\nu' = \Delta\nu/\sqrt{3}$ as above), in which case the dump intervals just calculated would result in a sensitivity loss of 3%. Note that the dump interval is proportional to $\sqrt{X}$, so to reduce the sensitivity loss further by a factor of three the dump intervals would be reduced by a factor of $\sqrt{3}$.

Consider phasing the array to provide a single output signal (equivalent to the "analog sum" of the VLA). Since, for the case under consideration, we cannot adjust the sampler timing, the finest delay steps are those provided by the sample interval. Suppose that the required bandwidth is reduced from the maximum of 2 GHz for any baseband signal by a factor β that is an integral power of two. Then if the sampling rate remains that for the full bandwidth, the signal is oversampled by a factor β. The time interval between samples is $1/2\Delta\nu$ where $\Delta\nu$ is the full bandwidth of 2 GHz. For the fine delay adjustment we pick the sample nearest the correct sample time, so the maximum delay error is $\pm 1/4\Delta\nu$. For each individual antenna the probability distribution of the delay errors is a rectangular (constant amplitude) function of width $1/2\Delta\nu$ centered on zero. Now consider one antenna of the MMA correlated with a signal from an antenna that is not part of the array, as in VLBI. The delay errors are those resulting from the finite delay resolution of the MMA. When the MMA is used in the phased array mode for VLBI, then the result can be considered as the sum of cross correlations of this type over the individual antennas of the MMA.

With delay error τ the correlator output for the case just outlined is proportional to

$$\frac{1}{2\Delta\nu/\beta}\int_{-\Delta\nu/\beta}^{\Delta\nu/\beta}\cos(2\pi\nu\tau)d\nu = \frac{\sin(2\pi\Delta\nu\tau/\beta)}{2\pi\Delta\nu\tau/\beta}. \tag{9}$$

Here it has been assumed that with the reduced bandwidth $\Delta\nu/\beta$ the signal at the sampler input has a baseband spectrum, that is, extending from near zero to $\Delta\nu/\beta$ in frequency. To obtain the average output as the delay changes by the interval $1/2\Delta\nu$ between samples we integrate the sinc function in Eq. (9) over the delay range 0 to $1/4\Delta\nu$:

$$4\Delta\nu\int_0^{1/4\Delta\nu}\frac{\sin(2\pi\Delta\nu\tau/\beta)}{2\pi\Delta\nu\tau/\beta}d\tau = \left(1 - \frac{x^2}{18} + \frac{x^4}{600} - \frac{x^6}{35,280} + \ldots\right), \tag{10}$$

where $x = \pi/2\beta$ and we have used the usual series expansion for the sine. (Tabulated values of the sine integral Si(x) could also be used to evaluate the integral). In the case where $\beta = 1$ the expression on the right hand side becomes $1 - 0.1371 + 0.0101 - 0.00043 + \ldots = 0.873$, so from the progression of the values one can see that the result is good to better than 0.1%. Thus with an array of MMA antennas in which the phased sum of the signals is correlated with a signal from an independent antenna, the loss in sensitivity due to MMA delay increments that correspond to the Nyquist sample rate is 12.7%, when the full bandwidth is used. Note that in terms of the sum of the phased signals this is the loss in *voltage* and the *power* loss is $1 - 0.873^2$ which is 23.8%. The Table below gives the results for several values of β.

β	Voltage Loss	Power Loss
1	12.7%	23.8%
2	3.4%	6.7%
4	0.86%	1.71%

Since there are eight baseband signals of width 2 GHz planned for each antenna of the MMA, the total bandwidth available in phased sum form is 8 GHz with $\beta = 2$, and 4 GHz with $\beta = 4$. These bandwidths would probably suffice for a long time into the future, so the need to operate in the phased array mode does not rule out the possibility of delay adjustment after correlation. The advantage of correcting for the delay errors after correlation is that a common 4 GHz clock could be used for all of the samplers required for the array, whereas with sampler phasing a separate phased clock is required for each antenna. However, the phased clock signals are technically not difficult to generate. One way to produce a phased clock would be to phase-lock a 4 GHz oscillator to a reference signal and insert the phase increments in the IF of the phase-locked loop. With a loop IF of, say, 10 MHz the IF reference with the required phase offsets could be generated digitally. Note that the additional hardware for phasing the sampler clock is required on a per-antenna basis, whereas the additional computing power for correction after correlation is required on a per-baseline basis.

Appendix

An equivalent way to approach the numerical result given in Eq. (10) is to consider the mean squared phase error. The rms delay error is

$$\Delta\tau_{\rm rms} = 1/4\sqrt{3}\Delta\nu, \tag{11}$$

where $\Delta\nu$ is the full bandwidth. For the case where the reduced bandwidth is in the form of a baseband with lowpass cutoff at $\Delta\nu/\beta$, rms value of the frequency is $\Delta\nu/\sqrt{3}\beta$. To obtain the rms phase error we multiply the delay error in Eq. (11) by 2π times the rms frequency, and obtain

$$\Delta\phi_{\text{rms}} = \pi/6\beta. \tag{12}$$

Then from Eq. (4) the sensitivity resulting from delay errors is $[1-\frac{1}{2}(\pi/6\beta)^2]$, which is identical to the first two terms on the right-hand side of (10), the other terms having been omitted in the cosine expansion in (4).

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