ALMA Memo 388
Degradation of Sensitivity Resulting from Bandpass Slope

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Abstract. The degradation in sensitivity resulting from a linear slope in the frequency response at the correlator input is calculated as a function of the decibel difference in the levels at the band edges. Three cases are considered, in which the slope is linear in voltage, linear in power, and linear in decibels. In all cases the dergradation is slightly less than 1% for a 2 dB difference across the band. An alternative approach to the calculation is given in the Appendix.

For a given IF bandwidth $\Delta\nu$ the sensitivity of an interferometer is maximized if the system gain is constant with frequency over the bandwidth, that is, the bandpass shape is rectangular. The responses of electronic components, including attenuation in cables, generally result in a deviation from the ideal bandpass shape. A major feature of this deviation is often a sloping response as in Fig. 1. Variations in bandpass shape result from the analog section of the system, and in the ALMA array, in which the signals are digitized at the antennas, one can expect the bandpass shapes to be similar for all antennas. This is unlike the original VLA system in which the waveguide responses, which are different for each antenna, are a major contributor to deviations of the bandpass responses. Closure errors can result from variations in the responses from one antenna to another, but if the deviations from the ideal rectangular shape are similar for all antennas, the major effect will be a degradation in sensitivity, which is the subject of this memorandum.

A derivation of the degradation factor resulting from the shape of an RF or IF bandpass function can be found in Thompson Moran and Swenson (1986, see Eq. 7.25; 2001, see Eq. 7.36). In the case where the frequency responses (in both amplitude and phase) are the same for both antennas, the signal to noise ratio is proportional to a degradation factor

$$\mathcal{D} = \frac{\int_0^\infty |h(\nu)|^2 d\nu}{\sqrt{\Delta\nu \int_0^\infty |h(\nu)|^4 d\nu}}, \tag{1}$$

where ν represents frequency and $h(\nu)$ is the *voltage* frequency response of the signals at the correlator input. Three cases are considered in which (1) the voltage response, (2) the power response, and (3) the response expressed in decibels, varies linearly with frequency.

Voltage-linear slope The voltage response $h(\nu)$ varies linearly with frequency as in Fig. 1. If h_0 is the voltage gain at the center frequency ν_0 we can write,

$$h(\nu) = (h_0 - \Delta h/2) + \Big(\nu - \nu_0 + \frac{\Delta\nu}{2}\Big)\frac{\Delta h}{\Delta\nu}, \qquad \nu_0 - \frac{\Delta\nu}{2} < \nu < \nu_0 + \frac{\Delta\nu}{2}. \tag{2}$$

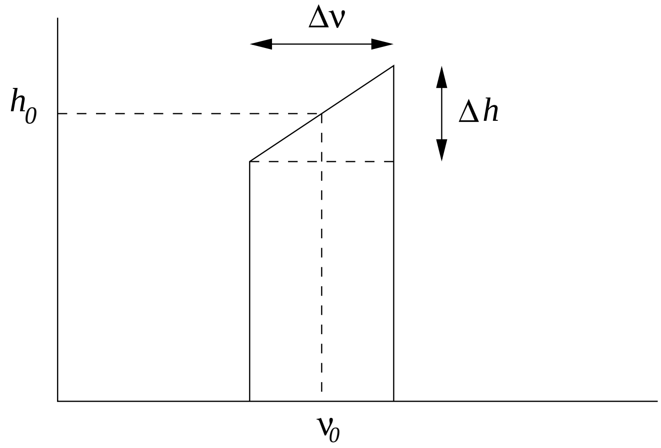

Figure 1: Frequency spectrum at the correlator input for center frequency ν_0 and bandwidth $\Delta\nu$. h_0 is the voltage level at the band center and Δh is the difference in levels between the band edges. Note that the calculated values of the degradation factor depend on the difference in power levels at the band edges, but not on the direction of the slope (i.e. whether the response increases or decreases with frequency).

Then,

$$\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} |h(\nu)|^2 d\nu = \Delta\nu\Big(h_0^2 + \frac{1}{12}\Delta h^2\Big), \tag{3}$$

and

$$\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} |h(\nu)|^4 d\nu = \Delta\nu\Big(h_0^4 + \frac{1}{2}h_0^2\Delta h^2 + \frac{1}{80}\Delta h^4\Big). \tag{4}$$

From Eq. 1, the degradation factor for this case is

$$\mathcal{D}_{\rm volt} = \frac{1 + \frac{1}{12}\left(\frac{\Delta h}{h_0}\right)^2}{\sqrt{1 + \frac{1}{2}\left(\frac{\Delta h}{h_0}\right)^2 + \frac{1}{80}\left(\frac{\Delta h}{h_0}\right)^4}}. \tag{5}$$

If δ is the difference in the gains at the band edges measured in decibels,

$$\delta = 10\log_{10}\left(\frac{1 + \frac{1}{2}(\frac{\Delta h}{h_0})}{1 - \frac{1}{2}(\frac{\Delta h}{h_0})}\right)^2, \quad \text{and} \quad \frac{\Delta h}{h_0} = \frac{2(10^{\delta/20} - 1)}{10^{\delta/20} + 1}, \tag{6}$$

which allows $\mathcal{D}_{\rm volt}$ to be calculated from given values of δ.

Power-linear slope The power response is $g(\nu) = |h(\nu)|^2$. The bandpass response is equivalent to that in Fig. 1 if we imagine that the symbols h_0 and Δh are replaced by g_0 and Δg, which are the mid-band power gain and the difference in the band edge power gains, respectively. In this case we have

$$g(\nu) = (g_0 - \Delta g/2) + \Big(\nu - \nu_0 + \frac{\Delta\nu}{2}\Big)\frac{\Delta g}{\Delta\nu}, \qquad \nu_0 - \frac{\Delta\nu}{2} < \nu < \nu_0 + \frac{\Delta\nu}{2}, \tag{7}$$

$$\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} |h(\nu)|^2 d\nu = \int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} g(\nu) d\nu = g_0 \Delta\nu, \tag{8}$$

and

$$\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} |h(\nu)|^4 d\nu = \int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} g^2(\nu) d\nu = \Delta\nu\Big(g_0^2 + \frac{1}{12}\Delta h^2\Big). \tag{9}$$

Thus from Eq. 1 we obtain for the degradation factor

$$\mathcal{D}_{\text{power}} = \frac{1}{\sqrt{1 + \frac{1}{12}\left(\frac{\Delta g}{g_0}\right)^2}}. \tag{10}$$

The relation between the decibel difference in the gains at the band edges and $\Delta g/g_0$ is given by

$$\delta = 10 \log_{10}\left(\frac{1 + \frac{1}{2}(\frac{\Delta g}{g_0})}{1 - \frac{1}{2}(\frac{\Delta g}{g_0})}\right), \quad \text{and} \quad \frac{\Delta g}{g_0} = \frac{2(10^{\delta/10} - 1)}{10^{\delta/10} + 1}, \tag{11}$$

which allows $\mathcal{D}_{\text{power}}$ to be calculated from δ.

Slope linear in decibels Here the bandpass response in Fig. 1 is taken to represent the power response measured in decibels. For the power response as a function of frequency we can write $h^2(\nu) = h_0^2 \exp[\sigma(\nu - \nu_0)]$. Then the required integrals are

$$\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} |h(\nu)|^2 d\nu = \frac{1}{\sigma}\Big[e^{\sigma\Delta\nu/2} - e^{-\sigma\Delta\nu/2}\Big], \tag{12}$$

and

$$\int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} |h(\nu)|^4 d\nu = \frac{1}{2\sigma}\Big[e^{\sigma\Delta\nu} - e^{-\sigma\Delta\nu}\Big]. \tag{13}$$

From Eq. 1, the degradation factor is

$$\mathcal{D}_{\text{dB}} = \sqrt{\frac{2(e^{\sigma\Delta\nu} - 1)}{\sigma\Delta\nu(e^{\sigma\Delta\nu} + 1)}}. \tag{14}$$

(This result can be found in Table 7.1 of Thompson et al. (1986, 2001) for the case where the slope parameter σ takes different values for the two antennas.) The decibel difference between the gains at the band edges is

$$\delta = 10 \log_{10}\left(\frac{e^{\sigma(\nu_0+\Delta\nu/2)}}{e^{\sigma(\nu_0-\Delta\nu/2)}}\right) = 10(\log_{10} e)\sigma\Delta\nu, \tag{15}$$

which allows $\mathcal{D}_{\text{dB}}$ to be calculated from δ.

Results Curves for the degradation as a function of the decibel difference δ (calculated using Mathcad) are given in Fig. 2 for the range of values which are of practical interest. Specification of a limit on δ for the ALMA receiving system requires consideration of the magnitude of other instrumental effects that result in degradation of sensitivity. In general, however, a tolerance on

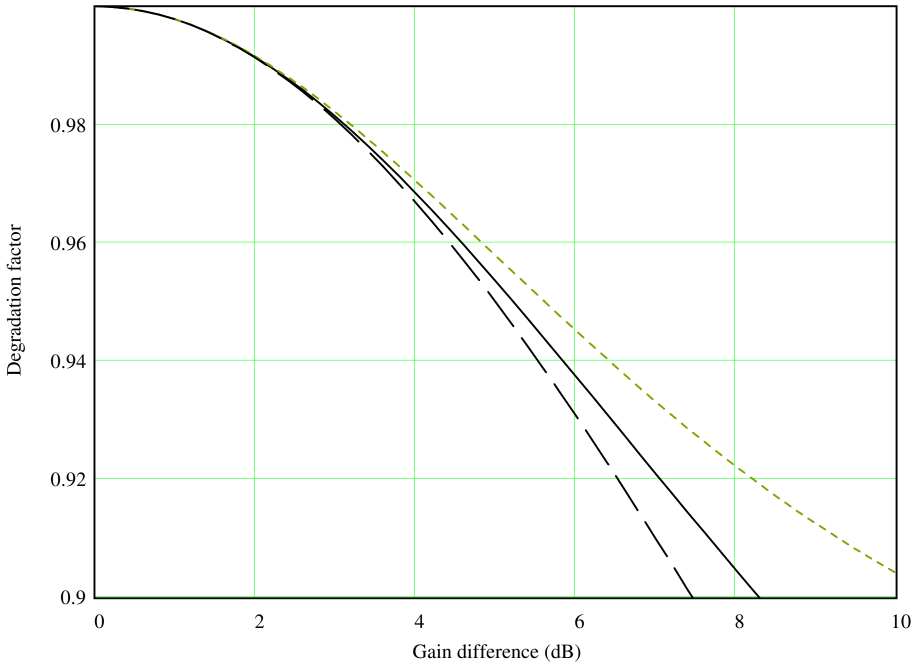

Figure 2: Degradation factor as a function of the gain difference in decibels at the band edges. The solid curve is the voltage-linear response, the small-dash curve is the power-linear response, and the large-dash curve is the decibel-linear response.

the slope of 2 dB across the band, which results in 0.86% degradation in sensitivity, is a desirable goal.

The behaviour of the degradation as δ becomes very large is of mathematical rather than practical interest. In the voltage-linear case the response becomes a triangular function and the degradation factor as a function of δ levels off at a value of 0.745. Similarly, in the power-linear case the degradation levels off at 0.866. In the decibel-linear case the degradation factor continues to decrease towards zero as δ increases, since the bandwidth at any decibel level is inversely proportional to δ.

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Appendix

An alternative approach, illustrated by derivation of $\mathcal{D}_{\mathrm{power}}$, is as follows. Consider an interferometer with an ideal rectangular passband of width $\Delta\nu$. The signal-to-noise ratio is $R_{\mathrm{sn}}(\Delta\nu)$. Now suppose that the bandwidth is divided into N channels of width $\Delta\nu/N$, as in a spectral-line correlator. The signal to noise ratios for the individual channels are all equal to $R_{\mathrm{sn}}(\Delta\nu)/\sqrt{N}$. Suppose that the IF *power* gains of the channels are not equal but are proportional to 1, $(1+\rho)$,

$(1+2\rho)$,... $[1+(N-1)\rho]$, in order of increasing frequency. The correlator outputs for the individual channels are summed without correction for the gain variation. The result is the interferometer response for a bandwidth with a stepped gain function. In summing the channel outputs, the signal voltages (from the correlator output) for the channels combine additively, and are each proportional to the IF power at the correlator input. Thus the combined signal is equal to the signal in the first channel multiplied by

$$\sum_{n=1}^{N}[1+(n-1)\rho] = N[1+\frac{1}{2}(N-1)\rho]. \tag{16}$$

The rms noise from each channel is proportional to the IF power gain, but the noise voltages at the detector outputs, being uncorrelated, combine as the square root of the sum of the squares. Thus the rms noise after combination is equal to the noise in the first channel multiplied by

$$\sqrt{\sum_{n=1}^{N}[1+(n-1)\rho]^2} = \sqrt{N\left[1+(N-1)\rho+\frac{(N-1)^2\rho^2}{3}+\frac{(N-1)\rho^2}{6}\right]}. \tag{17}$$

[In Eqs. 16 and 17 the following summations have been used: $\sum_{n=1}^{N} n = N(N+1)/2$ and $\sum_{n=1}^{N} n^2 = N(N+1)(2N+1)/6.0$.] From Eqs. (16) and (17) the signal to noise ratio of the combined channels is

$$\frac{R_{\rm sn}(\Delta\nu)}{\sqrt{N}} \times \frac{N[1+\frac{1}{2}(N-1)\rho]}{\sqrt{N\left[1+(N-1)\rho+\frac{1}{3}(N-1)^2\rho^2+\frac{1}{6}(N-1)\rho^2\right]}} \tag{18}$$

Now $(N-1)\rho$ is equal to the total increase in gain from the first channel to the last, which we will denote by $\rho_{\rm tot}$. Keeping $\rho_{\rm tot}$ constant, let N tend to infinity. Then ρ tends to zero and the stepped response becomes a smooth slope. The combined signal-to-noise ratio [expression (18)] becomes

$$R_{\rm sn}(\Delta\nu) \times \frac{1+\frac{1}{2}\rho_{\rm tot}}{\sqrt{1+\rho_{\rm tot}+\frac{1}{3}\rho_{\rm tot}^2}}. \tag{19}$$

The expression on the right-hand side of the multiplication symbol is the degradation factor, and is equal to unity for $\rho_{\rm tot}=0$. This result can be related to the decibel ratio of the gains at the band edges by $\delta = 10\log_{10}(1+\rho_{\rm tot})$. In terms of $\mathcal{D}_{\rm power}$ in Eq. 10, $(\Delta g/g_0)$ is equal to $\rho_{\rm tot}/[1+(\rho_{\rm tot}/2)]$. By substituting $\rho_{\rm tot} = (\Delta g/g_0)/[1-(\Delta g/2g_0)]$ in expression (19), the degradation factor is found to be identical to the result in Eq. 10.

Reference

Thompson, A. R., J. M. Moran, and G. W. Swenson, Jr., *Interferometry and Synthesis in Radio Astronomy*, Wiley, NY, 1986 (reprinted by Krieger), and 2001.