ALMA Memo 482
An Analysis on the Affect of Dual Wavelength Polarization Alignment on the Accuracy of Recovered Beat Note Phase

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1 Introduction

A status report [1] included an analysis of the the phase and amplitude due to the beating of two elliptically polarized light beams. In this report, the problem will be re-examined and analyzed in a more general way. The result will be used to investigate the sensitivity of the phase of the beat note response to mismatches of the polarization states of the two incident light beams.

2 Statement of the Problem

An elliptically polarized wave can be written in terms of the electrical field as: $E = [E_1 + j\,E_2 \exp(j\,\delta)] \exp(j\,\omega t)$, where E_1 and E_2 are (real) amplitudes of two arbitrary orthogonal polarizations used as the polarization description basis, δ is the phase angle by which E_2 leads E_1 and ω is the radian frequency of the light wave. The choice of polarization base is completely arbitrary. Without loss of generality, we can choose to use horizontally and vertically polarized waves as the basis, with the real component, E_1, used for the amplitude of the horizontally polarized portion of the wave and E_2 used for the amplitude of the vertically polarized portion of the wave. The above equation can be written as $E = [H + j\,V \exp(j\,\delta)] \exp(j\,\omega t)$ for reference.

The equations for two generally polarized light waves traveling down a fiber at two different frequencies, ω_1 and ω_2, are given by:

$$E_1 = (a_1 + j\,b_1 \exp(j\,\delta_1) \exp(j\,\omega_1 t) \quad (1)$$
$$E_2 = (a_2 + j\,b_2 \exp(j\,\delta_2) \exp(j\,\omega_2 t) \quad (2)$$

The analysis in [1] used an additional phase factor between the horizontal components of the two waves. This is an unnecessary complication. The implication of setting this phase to zero is that the beat note will have zero phase at time $t = 0$.

As in [1], we will now find the complex cross correlation between the two beams of light ($E_1 \cdot E_2^*$). We will choose to ignore the responsivity of the photomixer. If we want to include the photomixer responsivity later, we can follow the steps in [1]. Note, the complex cross correlation has units of power (neglecting constants of proportionality).

$$\begin{aligned} E_1 \cdot E_2^* &= (a_1 + j\, b_1 e^{j\,\delta_1}) e^{j\,\omega_1} (a_2 - j\, b_2 e^{-j\,\delta_2}) e^{-j\,\omega_2} && (3) \\ &= (a_1 a_2 + b_1 b_2 e^{j\,\delta} + j\, b_1 a_2 e^{j\,\delta_1} - j\, b_2 a_1 e^{-j\,\delta_2}) e^{j\,\omega_o \cdot t}, && (4) \end{aligned}$$

where we have defined $\delta = \delta_1 - \delta_2$ and ω_o ($\omega_o = \omega_1 - \omega_2$) is the beat note frequency. The asterisk superscript denotes complex conjugation.

After careful manipulation we find:

$$\begin{aligned} E_1 \cdot E_2^* &= (A + j\, B) e^{j\,\omega_o \cdot t} && (5) \\ &= A \cos(\omega_o t) - B \sin(\omega_o t) + j\, [B \cos(\omega_o t) + A \sin(\omega_o t)] && (6) \end{aligned}$$

where A and B are given by:

$$\begin{aligned} A &= a_1 a_2 + b_1 b_2 \cos\delta - a_2 b_1 \sin\delta_1 - a_1 b_2 \sin\delta_2 && (7) \\ B &= b_1 b_2 \sin\delta + a_2 b_1 \cos\delta_1 - a_1 b_2 \cos\delta_2 && (8) \end{aligned}$$

It is advantageous at this point to examine the phase of the cross correlation.

$$\begin{aligned} \Phi &= \arctan[\mathrm{Im}(\mathrm{E}_1 \cdot \mathrm{E}_2^*)/\mathrm{Re}(\mathrm{E}_1 \cdot \mathrm{E}_2^*)] && (9) \\ &= \arctan[(B \cos\omega_o t + A \sin\omega_o t)/(A \cos\omega_o t - B \sin\omega_o t)] && (10) \\ \Phi &= \arctan[(B/A + \tan\omega_o t)/(1 - B/A \tan\omega_o t)] && (11) \end{aligned}$$

In the limit, when B/A is zero,

$$\begin{aligned} \Phi &= \arctan[\tan\omega_o t] && (12) \\ &= \omega_o t && (13) \end{aligned}$$

and the phase of the cross correlation is identical to the beat note phase (see figures 1 and 2). For small values of B/A, the phase value is a biased version of the required phase. If anything in the system (such as the azimuth wrap) creates polarization changes that are different for the two light waves, then the phase of the beat note will change due to relative changes in the polarization states.

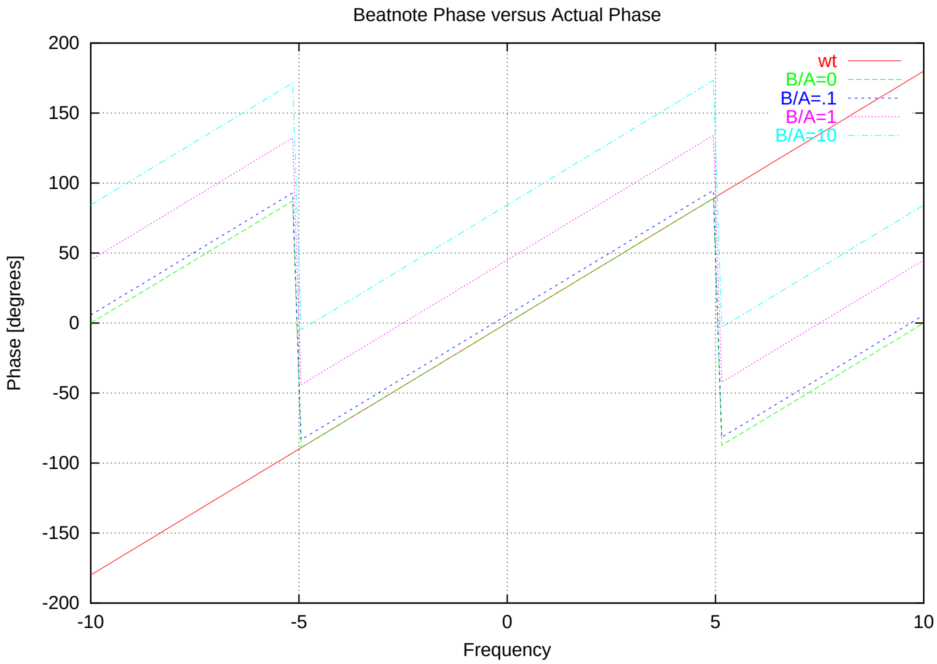

Figure 1: Example of how polarization mismatch causes a bias in the phase response.

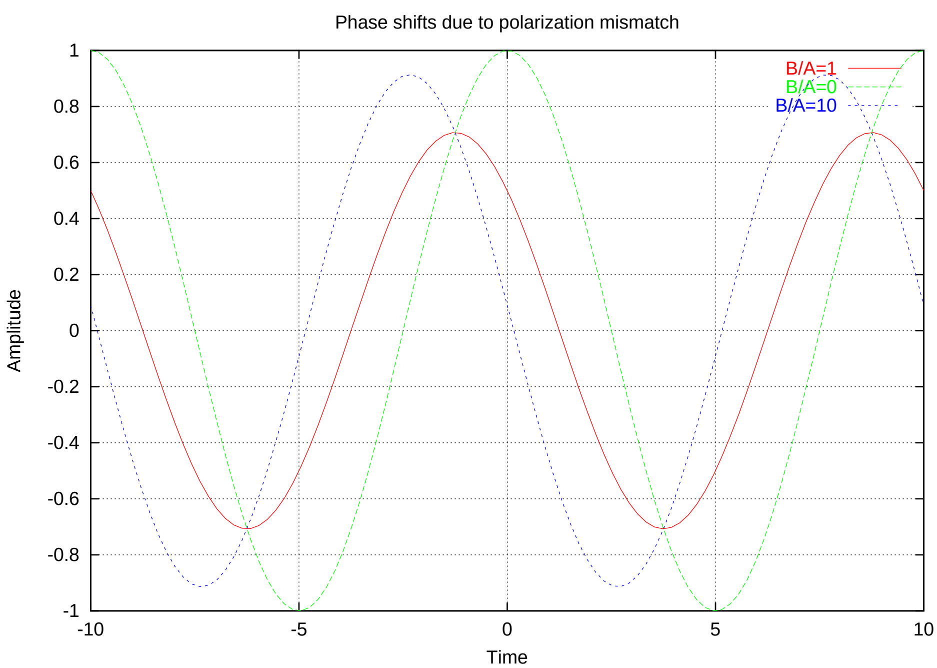

Figure 2: Example of how polarization mismatch causes a bias in the phase response.

3 Stokes Parameter Solution

Given that we have chosen the polarization basis for the wave description as orthogonal horizontal and linear components ($E = a + j\, b e^{j\,\delta}$), we can immediately write the polarization state in terms of the Stokes parameters.

$$I = |a|^2 + |b|^2 \quad (14)$$

$$Q = |a|^2 - |b|^2 \quad (15)$$

$$U = 2ab\cos\delta \quad (16)$$

$$V = 2ab\sin\delta \quad (17)$$

This is a general description and can be applied to each light wave.

We are interested in the solution to:

$$B = b_1 b_2 \sin\delta + a_2 b_1 \cos\delta_1 - a_1 b_2 \cos\delta_2 = 0 \quad (18)$$

All of the variables in this equation can be found through manipulation of the Stokes parameters. For example,

$$a_1 = \sqrt{\frac{I_1 + Q_1}{2}} \quad (19)$$

$$b_1 = \sqrt{\frac{I_1 - Q_1}{2}} \quad (20)$$

In terms of Stokes parameters, the solution to equation (18) is given by:

$$\frac{V_1 - I_1 - Q_1}{U_1} = \frac{V_2 - I_2 - Q_2}{U_2} \quad (21)$$

In terms of the polarizations state as defined by a, b and δ:

$$\tan(\delta_2) - \frac{a_2}{b_2\cos(\delta_2)} = \tan(\delta_1) - \frac{a_1}{b_1\cos(\delta_1)} \quad (22)$$

One obvious solution is for the polarization states to be perfectly matched, that is, $\delta_1 = \delta_2$ and $a_2/b_2 = a_1/b_1$. For any fixed polarization state for one frequency, there is a continuum of polarization states for the second frequency for which the phase of the cross correlation is equal to the phase of the beat note. This can be seen by picking a polarization state described by a_1/b_1 and δ_1 and finding a_2/b_2 as a function of δ_2 as δ_2 is swept over the range, $-\pi$ to π.

$$a_2/b_2(\delta_2) = [\tan\delta_2 - \tan\delta_1 + a_1/(b_1\cos\delta_1)]\cos\delta_2 \quad (23)$$

However, these other non-polarization matched solutions are not particularly important since any depolarization effects in the transmission system will, in general, depolarize different polarizations states differently. The only way to assure accurate phase is to guarantee an exact polarization match between the two light waves at the point the two light waves are combined onto a single fiber.

4 Polarization Transformations

The mathematical description of the transformation of polarization states involves bilinear fractional transforms; a type of conformal mapping. As in any conformal mapping there are always two states which transform to themselves [3][4]. These polarizations are known as characteristic polarization (CP) states or principal polarization states (PSP). In fiber terminology, polarization mode dependent (PMD) losses will transform polarization states along curves that connect the one CP with the other. Differential group delay (DGD) will transform polarization states around one of the CP states. For a given PMD and DGD, different polarization states will transform by different amounts. However, if two light waves start with matched polarization states, then any transformation of the polarization states will be the same for the two light waves as long as *the wave length difference is not too great*. The highest beat note frequency for ALMA is 142 GHz. That the polarizations remain matched through 15 km of fiber and various optical components needs to be verified emperically at the highest beat note frequency.

5 Polarization Match Factor

Reference [1] contains an equation to describe the polarization match factor between the two polarized waves. This equation is correct and is repeated here:

$$\cos(\beta) = s_1^A s_1^B + s_2^A s_2^B + s_3^A s_3^B \tag{24}$$

where β is the angle between the two polarization states on the Poincaré sphere (Stokes sub-space), A and B designate the two light waves and the s_1, s_2 and s_3 represent the normalized Stokes parameters $Q/I, U/I$ and V/I, respectively.

Reference [2] gives a similar equation except it is for two antennas operating in a transmit/receive mode,$\cos(\beta) = s_1^A s_1^B - s_2^A s_2^B + s_3^A s_3^B$. A polarization match factor is also given:

$$\rho = \cos^2\left(\frac{\beta}{2}\right) \tag{25}$$

$$0 \leq \rho \leq 1 \tag{26}$$

$\rho = 1$ for two waves that are fully polarization matched and $\rho = 0$ for two waves that are cross polarized.

6 Conclusions

A figure in [1] shows how the mock antenna azimuth wrap transforms an input polarization state into a different output polarization state depending on the azimuth position. This polarization transformation will cause the recovered beat note phase to be different than the actual beat note phase unless the two light wave polarization states are matched at the input to the system. When the two light waves have matched polarization states, any change in the polarization state due to system effects, such as azimuth wrap, will have no affect on the accuracy of the recovered beat note phase.

The beat note can be phase shifted due to physical changes in the electrical line length of the fiber *and* changing mismatches in the polarization states of the two light beams. The Line Length Correction systems is only capable of correcting for the former effect. It is important then to maintain a polarization match between the two light beams to prevent phase shifts in the beat note which cannot be corrected.

The same considerations need to be made on the antenna side, where the signal is frequency shifted and sent back down the fiber to complete the round trip for phase correction.

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7 References

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